

# Accuracy, Noise, and Scalability in Quantum Computation: Strategies for the NISQ Era and Beyond

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Received: 26.05.2025

## “Article 3 of the series”

**Abstract:** Quantum computers promise to revolutionize science and technology by offering the potential to solve complex problems intractable with classical approaches. However, realizing this potential hinges on effectively managing the noise and errors inherent in quantum systems, which threaten computational accuracy. This work (Accuracy, Noise, and Scalability in Quantum Computation [Unpublished doctoral dissertation IV. Report]. Gebze Technical University, Kocaeli, Türkiye [318, 462]) has explored a broad spectrum, from the fundamentals of quantum computation to strategies for enhancing the performance of devices in the Noisy Intermediate-Scale Quantum (NISQ) era, with a particular focus on the critical role of quantum error correction (QEC) codes and the decoder algorithms developed for them. While various methods exist for characterizing and manipulating quantum states, the scalability of these methods becomes a significant issue as the number of qubits increases. The measurement process itself also requires careful planning as it perturbs the quantum state. QEC codes, especially topological codes like surface codes, developed to overcome these challenges, form the foundation of fault-tolerant quantum computation. The success of a QEC code largely depends on the performance of its decoder algorithm, which analyses error syndromes to detect and correct the most probable errors. Alongside classical approaches like Minimum-Weight Perfect Matching (MWPM) and Union-Find, newer and potentially more powerful methods such as Maximum-Likelihood Decoders (MLD) and Neural Network-based Decoders (NNbD) are active areas of research. A prominent aspect of this study is the demonstration that, even with limited classical computing resources, the theoretical scalability of quantum error correction mechanisms can be pushed to remarkable limits using sophisticated simulation techniques and algorithmic ingenuity. Notably, striking results such as the simulation and verification of surface code error correction algorithms for systems of 25 million theoretical qubits have been achieved on a personal computer. Furthermore, the graphical visualization of error correction solutions for systems exceeding 100,000 theoretical qubits underscores the analysability of such complex systems. These findings indicate that error correction principles are theoretically applicable to very large systems and that classical simulations continue to be a valuable tool in this exploratory journey. In the future, key objectives will include the development of more efficient and scalable decoders, the discovery of new QEC codes, the creation of realistic noise models, advancements in hardware-software co-design, and the execution of complex algorithms on logical qubits. Quantum error correction will continue to play a central role on the path to fault-tolerant quantum computation, and theoretical and simulation-based work in this area will offer significant contributions to the realization of practical quantum computers. Large-scale simulation achievements driven by the creativity of individual researchers, as highlighted here, bolster hopes for the future of the field.

**Keywords:** Quantum Computing, Decoder, Simulation, Scalability, Qubit, Quantum Error Correction, QEC, Stabilizer Codes, Topological Codes, Surface Code, Fault-Tolerant Quantum Computation, Quantum Noise.

**Note:** Citations and numbering are in continuation of the previous article.

## I. Strategies for Enhancing Accuracy in Quantum Computation: Characterization, Scaling Challenges, and Noise Management

The potential of quantum computers to solve complex problems intractable for classical supercomputers has generated immense excitement in the scientific and technological communities. However, fully realizing this potential depends on overcoming fundamental challenges, particularly in increasing accuracy rates and ensuring system scalability. The pursuit of practical and powerful quantum computers necessitates a relentless focus on improving the accuracy of quantum operations and computations. While the theoretical underpinnings of quantum mechanics offer a pathway to solving problems intractable for classical machines, the inherent fragility of quantum states and the complexities of controlling quantum systems present significant hurdles. A multitude of methods are being developed and applied to bolster these accuracy rates, each with its own strengths and limitations, particularly as systems scale to higher qubit counts.

### Quantum State Characterization and Verification Techniques

Understanding and verifying the state of a quantum system is a prerequisite for trustworthy quantum computation. Several prominent techniques are employed for this purpose:

- **Quantum State Tomography (QST):** This is a comprehensive diagnostic procedure aimed at fully reconstructing the quantum state (represented by its density matrix) of a system. QST involves performing a series of different measurements on identically prepared quantum states and then using statistical post-processing to infer the most likely state that would produce the observed measurement outcomes. While QST provides a complete description, it suffers from a severe scalability issue: the number of measurements and the computational resources required for post-processing grow exponentially with the number of qubits. Consequently, for systems exceeding approximately 10–15 qubits, full QST becomes practically infeasible, and for systems with more than, say, 50 qubits [185], it is currently an insurmountable challenge to obtain desired, fully characterized results with high fidelity. This limitation stems directly from the exponential growth of the Hilbert space dimension ( $2^n$  for  $n$  qubits), which necessitates an exponential number of parameters to describe an arbitrary quantum state.
- **Classical Shadow Tomography:** As a more scalable alternative to full QST, classical shadow tomography has emerged. This technique aims to predict many properties of a quantum state (e.g., expectation values of local observables, fidelities with target states) from a significantly smaller number of measurements. It involves applying random unitary transformations (often drawn from a specific distribution like random Pauli or Clifford unitaries) to the quantum state before measuring in a fixed basis. By collecting these "classical shadows," one can efficiently estimate various features of the underlying quantum state without needing to reconstruct the full density matrix. While not

providing a complete state description, it offers a powerful and resource-efficient tool for specific characterization tasks.

- **Black-Box Verification and Validation (QBVV):** In many practical scenarios, fully characterizing a quantum device or a complex quantum state is unnecessary or too costly. Black-box approaches focus on verifying whether a quantum device performs a specific task correctly or whether a prepared state meets certain criteria, without needing to know the intricate details of the system's internal state. This can involve randomized benchmarking, process tomography (for characterizing quantum operations rather than states), or specialized verification protocols.
- **Oracle-Guided Methods (or "Expert" Systems):** The term "oracle" in quantum computation typically refers to a black box that implements a specific function, often used as a subroutine within a larger quantum algorithm. In the context of characterization or accuracy enhancement, an "expert" system or oracle could represent a highly optimized classical simulation, a pre-calibrated reference, or a known theoretical model against which the quantum system's behaviour is compared. (Translating "oracle" as "expert" or "reference system" in this specific context is indeed more meaningful than "soothsayer"). These methods can guide the calibration process or help identify discrepancies between expected and observed behaviour.

**The Exponential Scaling Wall:** A common thread among many of these characterization methods is the challenge posed by the **exponential scaling problem**. The representation of many-body quantum states requires a complex matrix (the density matrix) whose dimensions grow exponentially with the number of particles (qubits). While this vast state space is precisely what empowers quantum technologies with their potential computational advantages, it simultaneously makes the task of fully characterizing, simulating, or verifying these systems exceedingly difficult beyond a modest number of qubits. Overcoming this scaling wall is a central research theme, necessitating the development of more efficient, approximate, or targeted characterization techniques.

### The Impact of Measurement and the Imperative for Estimation

A fundamental tenet of quantum mechanics is that **every time we measure a quantum system, its state collapses** into one of the eigenstates corresponding to the measurement observable. This measurement-induced collapse is an irreversible process that generally disturbs the quantum state, limiting the information that can be extracted from a single measurement and preventing continuous observation of the system's evolution. This has profound implications for computation and characterization:

- **Minimizing Measurements:** The necessity arises to perform measurements as infrequently as possible during a quantum computation to preserve coherence and quantum correlations. For characterization, this means that multiple identical preparations of the state are often needed to gather sufficient statistics.

- **Estimation over Direct Observation:** Since we cannot continuously observe the quantum state without destroying it, we often need to **find or infer the system's properties by estimation**, based on a limited set of measurements performed on an ensemble of identically prepared systems. This is where statistical inference and techniques like QST or shadow tomography play their roles.

**Measurement Calibration and Error Reduction:** The physical process of measurement is not perfect and is itself a source of error. **Measurement calibration** is crucial to reduce these "readout errors." This involves characterizing the probabilities of, for example, measuring a '0' when the qubit was actually in state '1', and vice-versa. By applying correction matrices derived from this calibration, the **probabilities of average outcomes can be more accurately determined**, leading to more reliable results from quantum experiments and computations.

### The Pervasive and Complex Nature of Noise

Beyond measurement-induced collapse and readout errors, quantum systems are constantly interacting with their environment, leading to **noise**. The primary effect of noise is to **produce incorrect or unintended outputs** by corrupting the delicate quantum states. The **overall effect of noise generated throughout a quantum computation is generally quite complex** and multifaceted:

- **Decoherence:** Noise causes qubits to lose their quantum properties (superposition and entanglement) over time, a process known as decoherence. This leads to a decay towards classical-like states.
- **Gate Errors:** Physical implementations of quantum gates are imperfect, leading to deviations from the ideal unitary operations they are intended to perform.
- **Crosstalk:** Unwanted interactions between different qubits or control lines can lead to correlated errors.

To accurately analyse and mitigate the impact of noise, one must consider how each gate and each potential error source transforms the quantum state and how these effects propagate and accumulate throughout the computation. This often requires sophisticated noise models and characterization techniques (like Quantum Process Tomography for gates, or Randomized Benchmarking).

**Noise in the Final Measurement:** It is crucial to recognize that noise occurs even in the final measurement stage, beyond the intrinsic collapse. The measurement apparatus itself can be noisy, or environmental noise can affect the qubits during the measurement process. Therefore, each measurement, in a noisy system, not only collapses the state but can also effectively increase the error rate by misinterpreting the collapsed state or by allowing further decoherence before the outcome is registered.

### Basis States and Idealized Assumptions for Error Analysis

When analysing a quantum system of  $n$  qubits, we often consider its state in terms of the computational basis, which consists of  $2^n$  possible basis states (e.g.,  $|00\dots0\rangle$ ,  $|00\dots1\rangle$ , ...,  $|11\dots1\rangle$ ). The aim



in many measurement schemes is to arrange the  $n$  qubits we will measure such that their final state can be projected onto this basis, yielding a classical bit string (e.g., "0110...").

For the purpose of isolating and understanding specific types of errors, we can initially assume that a measured gate (or the state preparation) is, hypothetically, error-free and noiseless. Under such an idealized assumption, if we then introduce a known measurement error model, it would not be overly difficult to determine the impact of these specific measurement errors, and we could precisely calculate the probabilities of observing different output bit strings. This "divide and conquer" approach helps in building more comprehensive error models by first understanding individual error components before combining them. However, in reality, all sources of error are often intertwined.

This detailed understanding of characterization methods, the implications of measurement, the pervasiveness of noise, and the challenges of scaling forms the bedrock upon which strategies for building fault-tolerant quantum computers are developed. The journey involves a continuous interplay between theoretical modelling, experimental characterization, and the development of sophisticated error mitigation and correction techniques.

In the current era, termed "Noisy Intermediate-Scale Quantum" (NISQ), quantum devices are being developed that do not yet possess full fault-tolerant error correction capabilities but can offer advantages over classical methods for specific problems. Intensive research is underway on various strategies and methods to maximize the performance of these devices.

## 1. Quantum State Characterization and Accuracy Enhancement Methods

Accurately determining and manipulating the states of qubits (quantum bits), which are fundamental to the operation of quantum computers, is critically important for the reliability of computations. Methods developed for this purpose include:

- **Quantum State Tomography (QST):** A technique aimed at fully reconstructing the state (density matrix) of a quantum system. QST estimates the quantum state of qubits by performing numerous different measurements on the system and statistically processing these measurement outcomes. It is a valuable diagnostic tool for understanding the initial state of the system, verifying the effect of quantum gates, and characterizing noise models. However, the major drawback of QST is that the required number of measurements and computational complexity increase exponentially with the number of qubits. Consequently, performing full QST for systems with more than 50 qubits is practically unfeasible with current resources. This stems from the fact that the dimension of the Hilbert space for many-body quantum systems grows exponentially (as  $2^n$ , where  $n$  is the number of qubits). Fully characterizing this vast Hilbert space requires determining an exponential number of parameters.

- **Classical Shadow Tomography:** A more efficient method developed as a solution to the exponential cost problem of full state tomography. Classical shadows significantly reduce the number of measurements needed to predict certain properties of a quantum system (such as expectation values of observables). After applying randomly chosen unitary transformations to the system, measurements are made in a standard basis, and from these measurements, "shadows" of the system are constructed. These shadows can be reused to predict many different properties of the system.
- **Black-Box Modelling and Oracle Usage:** Black-box approaches aim to model the system based on its input-output behaviour without detailed knowledge of its internal workings. The concept of an oracle, in quantum algorithms, refers to a hypothetical component that can compute a specific function flawlessly and instantaneously. In real-world applications, these oracles can be optimized classical or quantum routines that solve specific sub-problems. Such approaches can be used for targeted information extraction, especially in complex systems, rather than full characterization.

Most of these methods face challenges in terms of accuracy and efficiency, especially when dealing with a high number of qubits. The exponential scaling problem—the exponential increase in system complexity and resources required for characterization with an increasing number of qubits—is one of the biggest obstacles for quantum technologies. New and more efficient approaches are continually being developed to overcome this hurdle.

## 2. The Measurement Process, Noise, and Their Implications

One of the fundamental postulates of quantum mechanics is that the state of a quantum system collapses upon measurement. A qubit exhibiting quantum properties like superposition or entanglement will probabilistically collapse to a specific classical state (0 or 1) when measured. This collapse limits the complete information we can have about the system before measurement and is an irreversible process. Therefore, in quantum algorithms, measurements are typically performed at the very end of the computation and as sparingly as possible. This necessitates inferring intermediate states of the system through theoretical models and indirect deductions rather than direct observation.

- **Measurement Calibration:** Real-world measurement devices are not perfect and can introduce their own errors into the measurement process (readout errors). Measurement calibration aims to characterize these errors and correct the measurement outcomes to obtain more accurate probability distributions. For instance, instances where a qubit prepared in the "0" state is read out as "1" with a certain probability can be identified and corrected.
- **The Pervasive Impact of Noise:** Noise encompasses any random influence that disturbs the delicate nature of quantum states, arising from unwanted interactions of quantum systems with their environment or imperfections in control systems. Noise causes qubits to lose their quantum properties (coherence) and behave more like classical bits (decoherence). The cumulative effect of noise throughout a computation is quite complex. Each quantum gate (operation) and each qubit can be exposed to noise differently, and

this noise can be transformed and propagated by subsequent operations. Therefore, accurately modelling and mitigating the effects of noise requires a detailed understanding of how each gate and error transforms the quantum state. Noise (readout errors) can occur even at the final measurement stage, meaning each measurement can potentially increase the error rate.

- **State Space and Measurement Bases:** A system of  $n$  qubits is defined in a Hilbert space composed of  $2^n$  (two to the power of  $n$ ) possible basis states (computational basis states). In an ideal scenario, assuming a measured gate or qubit is error-free and noiseless, it would be easier to determine measurement errors and expected probability distributions. However, deviations from this ideal state are always present in real systems.

### 3. Noise Sources and Management in NISQ Devices

The defining characteristic of NISQ devices is their "noisiness." This noise can originate from various sources:

- **Rogue Qubits:** These are qubits that exhibit undesirable behaviour and are difficult to control due to design flaws, manufacturing defects, or environmental interactions. Rogue qubits can severely degrade computational accuracy by engaging in unwanted interactions with other qubits (crosstalk) or by increasing the overall noise level. Identifying them and mitigating their effects is crucial for improving the performance of NISQ devices.
- **Decoherence Mechanisms:** Noise randomly alters the state of a qubit (represented as a vector on the Bloch sphere), causing it to transition from a pure quantum state (on the surface of the sphere) to a mixed state (inside the sphere) or eventually collapse towards a classical bit state (near the poles). This process is called decoherence and is characterized by  $T_1$  (energy relaxation time) and  $T_2$  (phase coherence time).
- **Phase and Amplitude Information:** The state of a qubit contains both amplitude ( $|\alpha|^2$ ,  $|\beta|^2$ ) and phase ( $\phi$ ) information. During quantum computations, phase information allows qubits to create constructive and destructive interference, enabling complex algorithms. However, a standard measurement in the computational basis (typically the Z-basis) does not directly yield phase information; only the probabilities of finding the qubit in the  $|0\rangle$  or  $|1\rangle$  state (squares of the amplitudes) are measured. The effective angle in measurement is  $\theta$  (the polar angle on the Bloch sphere), and the measurement outcome of a qubit in superposition (e.g.,  $\theta=90^\circ$  for an equal superposition) is randomly determined at the poles with a 50:50 probability. This probability is given by the formula  $\cos^2(\theta/2)$ . For instance, for  $\theta=90^\circ$ ,  $\cos^2(45^\circ) \approx 0.5$ . This inherent probabilistic nature (randomness) of quantum mechanics means measurement outcomes are intrinsically uncertain, a fundamental difference from classical deterministic computations. This randomness can be beneficial for some applications, like quantum random number generators, while being a challenge to manage in algorithms expecting deterministic outcomes.

### 4. Advanced Error Management and Algorithmic Strategies

The inevitable presence of noise necessitates sophisticated strategies to ensure the reliability of quantum computations.

- **Entanglement and Control:** To limit the effects of rogue or decohered qubits, approaches such as entangling these qubits with auxiliary qubits (ancilla qubits) to indirectly monitor or correct their states are being investigated. However, how to efficiently implement such control mechanisms for a large number of qubits and how to design the necessary algorithms (sequences of quantum gates to manipulate qubit states) remain active research topics. Some methods effective for low qubit counts may not scale directly to high qubit numbers due to increased system size and complexity.
- **Probabilistic Nature of Algorithms:** Most quantum algorithms, especially in the NISQ era, do not guarantee 100% certainty. Instead, they aim to provide the correct result with high probability. Therefore, running the algorithm multiple times and statistically analysing the results is a common practice. When and how to perform measurements is a critical decision for the overall success of the algorithm.
- **Considerations in Algorithmic Design:** Some quantum algorithms, like Grover's search algorithm, promise speedups (e.g., quadratic for Grover's) over classical algorithms for certain problems. However, each step of these algorithms delicately changes the quantum state. In the presence of noise, these steps may not execute as planned, and the algorithm's performance can degrade. Therefore, designing algorithm steps to be noise-resilient or integrating error mitigation techniques to compensate for noise effects is important.

## 5. Quantum Measurement, Encoding, and Future Perspectives

Quantum measurement (or readout) is the fundamental physical process for transferring information encoded in quantum systems to the classical world. The most common type of measurement is performed in the computational basis (typically the Z-basis, representing  $|0\rangle$  and  $|1\rangle$  states) and is sufficient for many quantum computation tasks.

- **From Physical to Logical Qubits: The Journey to Error Correction:** Existing physical qubits are inherently very sensitive to noise and environmental influences, making them "impractical" for reliable, large-scale computation. To overcome this, quantum error correction (QEC) codes have been developed. These codes protect a more robust "logical qubit" from errors in individual physical qubits by distributing (encoding) information across multiple physical qubits. Physical qubits are often arranged in a lattice or similar topology, and interactions between them are used to encode, manipulate, and detect/correct errors (decoding and error correction) in the logical qubit's state. This process aims for information processing with minimal loss.
- **The Need for Software and Algorithm Development:** Implementing quantum error correction and error mitigation strategies requires sophisticated software modules and algorithms. However, the licensing of

some advanced software in this field or its specificity to particular hardware platforms can slow down the development of open-source and platform-independent solutions. Existing open-source tools are often not perfect in all aspects and can suffer from significant performance degradation, especially when simulating and analysing systems with a high number of qubits. This is also due to the exponential scaling problem limiting the ability of classical computers to simulate quantum systems.

- **Surface Codes and the Future:** Surface codes are considered one of the most promising QEC codes for building fault-tolerant quantum computers due to their high error threshold and applicability in a 2D planar architecture. New algorithms based on surface codes and optimizations to enhance their efficiency are critically important for enabling large-scale quantum computations. In this context, as thesis proposes, the development of a computational software module focused on decoding encoded quantum information with minimal damage could fill a gap in this area and make a significant contribution towards practical quantum computation.

In conclusion, realizing the promise of quantum computers requires a multidisciplinary effort across hardware, software, and algorithmic domains. Increasing accuracy rates, effectively managing noise, and developing scalable systems are fundamental research areas that will shape the future of this exciting technology.

## II. Qubit States, Measurement Mechanisms, and Error Correction Decoders

Understanding the behaviour of qubits, the fundamental building blocks of quantum computation, and their interaction with measurement processes is essential for fully grasping the potential of quantum systems. This chapter will explore the general superposition state of a single qubit, measurement probabilities within the framework of Born's rule, and the role of the phase angle in these processes. Subsequently, the inevitable environmental interactions and noise sources affecting physical qubits will be addressed, emphasizing the importance of quantum error correction codes and, particularly, the decoder algorithms developed for these codes.

### 1. The General State of a Qubit and Probability Amplitudes





Figure 29: Qubit processes

Unlike classical bits, the general state of a quantum bit (qubit) is not limited to being solely in the  $|0\rangle$  or  $|1\rangle$  state. Thanks to the superposition principle of quantum mechanics, a qubit can be expressed as a linear combination of these two basis states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad (41)$$

Here,  $\alpha$  and  $\beta$  are the probability amplitudes for the qubit being in the  $|0\rangle$  and  $|1\rangle$  states, respectively. These amplitudes are generally complex numbers, encoding both the magnitude and phase information of the qubit's quantum state. When a qubit is measured, this superposition state probabilistically collapses to one of the basis states. The probability of finding it in state  $|0\rangle$  is  $P(0) = |\alpha|^2$ , and in state  $|1\rangle$  is  $P(1) = |\beta|^2$ . The condition of normalization dictates that the sum of these probabilities must always be unity (equal 1):

$$|\alpha|^2 + |\beta|^2 = 1 \quad (42)$$

This equality ensures that the probabilities represent 100% of the possibilities and holds even when  $\alpha$  and  $\beta$  are real numbers, though they are typically complex. As special cases:

This ensures that the total probability is 100%. Special cases include:

- If  $\alpha=1$ ,  $\beta=0$ , then  $|\psi\rangle = |0\rangle$ , and the qubit is definitively in the  $|0\rangle$  state.
- If  $\alpha=0$ ,  $\beta=1$ , then  $|\psi\rangle = |1\rangle$ , and the qubit is definitively in the  $|1\rangle$  state.

A qubit's state can also be visualized geometrically on the Bloch sphere, a unit sphere in three dimensions. In this representation, the qubit state is defined by angles  $\theta$  and  $\phi$ :

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \quad (43)$$

Here,  $\theta$  (polar angle) is the angle the qubit's state vector makes with the +z-axis ( $|0\rangle$  state) on the Bloch sphere ( $0 \leq \theta \leq \pi$ ), and  $\phi$  (azimuthal angle) is the angle of the state vector's projection onto the xy-plane with respect to the +x-axis ( $0 \leq \phi < 2\pi$ ). For instance, the  $|+\rangle$  state, an eigenstate of Pauli-X operator located on the x-axis, corresponds to  $\theta = \pi/2$  and  $\phi = 0$ , yielding  $\alpha = \cos(\pi/4) = 1/\sqrt{2}$  and  $\beta = \sin(\pi/4) = 1/\sqrt{2}$ , thus  $|+\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$ .

From the probability amplitudes, if  $|\alpha|$  is greater than  $|\beta|$ , it indicates a higher probability of the system collapsing to the  $|0\rangle$  state upon measurement. Conversely, if  $|\alpha|$  is smaller than  $|\beta|$ , it implies a higher probability of collapsing to the  $|1\rangle$  state. This fundamental principle is consistent with Born's rule of quantum mechanics. Born's rule [186], formulated by Max Born, states that the probability of a quantum system's measurement yielding a particular outcome is proportional to the square of the absolute value (magnitude) of the wave function component (probability amplitude) corresponding to that outcome. The  $e^{i\phi}$  term in Equation 43 indicates that the phase angle ( $\phi$ ) enters the qubit's state description via complex numbers. In a standard computational basis measurement (typically a Z-basis measurement), the probabilities of the qubit collapsing to  $|0\rangle$  or  $|1\rangle$  ( $P(0) = \cos^2(\theta/2)$ ,  $P(1) = \sin^2(\theta/2)$ ) depend directly on the angle  $\theta$ . The phase angle  $\phi$  does not directly affect the collapse probabilities in such a single-qubit measurement; however, it plays a crucial role in interference patterns between qubits, in the application of quantum gates, and thus in the overall progression of a quantum algorithm.

## 2. Noise, Environmental Interactions, and the Role of Quantum Error Correction

It is an unavoidable reality that a physical qubit, contrary to the assumption of an ideal isolated system, interacts with its environment. Since it is practically impossible to fully control and isolate these environmental interactions, unwanted changes occur in the qubit's quantum state. These changes can arise from various noise sources. Unlike classical bits, a quantum bit (qubit) is not confined to deterministic '0' or '1' states. Due to the principle of superposition, a cornerstone of quantum mechanics, a qubit can exist as a linear combination of these two computational basis states, weighted by complex coefficients (41-43).

### 1. The Detrimental Effects of Noise and the Necessity of Quantum Error Correction

Contrary to idealized isolated quantum systems, physical qubits inevitably interact with their environment. These interactions lead to unwanted and often random changes (errors) in the qubit's delicate quantum state, a phenomenon known as decoherence, which causes the loss of quantum information over time. Major sources of noise include:

- **Random Couplings:** Unwanted interactions of the qubit with other systems, such as neighbouring qubits, control lines, or stray electromagnetic fields.

- **Environmental Noise Fluctuations:** Uncontrollable environmental factors like temperature variations, magnetic field fluctuations, or mechanical vibrations.
- **Imperfect Quantum Gates:** Physical mechanisms implementing quantum operations are not ideal, leading to deviations from the intended transformation.
- **Readout Errors:** The measurement process itself can be faulty, leading to an incorrect classical bit value being read out instead of the qubit's true state.

These noise sources are considered the "Achilles' heel" of quantum computation and are among the most significant obstacles to building large-scale, fault-tolerant quantum computers. Despite the random nature of noise, quantum error correction (QEC) codes assume that these errors have a certain structure and can be managed statistically. The fundamental principle of QEC is to create redundancy by distributing the information of a single logical qubit across many physical qubits (encoding). This way, errors occurring on a subset of physical qubits can be detected and corrected without corrupting the overall state of the logical qubit. "Quantifiable knowledge of errors" (i.e., characterization of the error model) and "measurement instruction for syndrome extraction" (i.e., how to extract error syndromes) form the basis of QEC protocols.

These situations should not be seen as dilemmas or unsolvable problems. Instead, many of the challenges, such as noise and decoherence, stem from the fundamental nature of quantum mechanics—the ontological structure of the universe at the microscopic level. Understanding and managing these natural processes is one of the primary motivations in the development of quantum technologies.

## 2. Quantum Error Correction Decoders: Mechanisms and Classification

When a QEC code is implemented, errors occurring on physical qubits generate an information pattern called the "error syndrome." This syndrome is typically obtained using auxiliary (ancilla) qubits and without directly measuring (and thus collapsing) the state of the logical qubit. The decoder takes this error syndrome as input, infers the most likely error configuration, and determines the correction operation (usually a combination of Pauli X, Y, Z operations) that needs to be applied to reverse the error.

## 3. Decoding Algorithms and Surface Codes

Pre-measurement processes, especially error correction mechanisms, are vital for ensuring the reliability of quantum computers.



Figure 30: Pre-measurement processes

In general, decoding algorithms are designed to identify errors that occur in quantum error correction codes and determine the corresponding correction operations. These algorithms can be generalized to various surface codes (with or without boundaries), toric codes (e.g., TOR(k) [183, 187, 188]), and color codes [159, 189, 190]. These codes protect a logical qubit by distributing information across multiple physical qubits. Decoders are a critical component for restoring the state of this logical qubit.

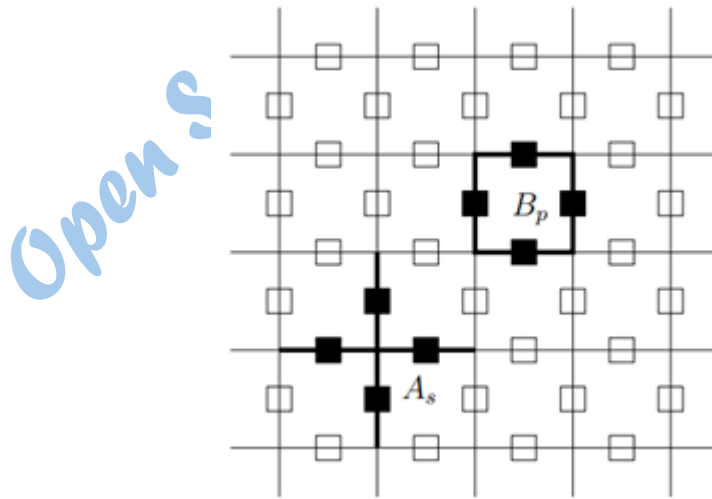


Figure 31: Toric code (TC) [191]. Spins are at the vertices of a square lattice.  $A_s$ : Star operator  $B_p$ : Plaquette operator.

The toric code (Figure 31) and surface codes are prominent examples of topological QEC codes, favoured for their high error thresholds (the limit of the physical error rate below which the logical error can

be suppressed) and compatibility with 2D planar architectures. In these codes, qubits are arranged on a lattice, and errors are detected by changes in the outcomes of local stabilizer measurements (the error syndrome).

$$\text{Hamiltonian: } H_T = -J_e \sum_s A_s - J_m \sum_p B_p \quad (44)$$

The toric code is a prototypical example of a topological quantum error correction code. As shown in Figure 31, qubits (spins) are typically placed on the edges or vertices of a square lattice. Error detection is performed by measuring local stabilizer operators, termed star ( $A_s$ ) and plaquette ( $B_p$ ) operators. These operators consist of products of Pauli X or Pauli Z on neighbouring qubits. The Hamiltonian of the toric code (Equation 44) is the sum of these stabilizer operators, and the ground state (lowest energy state) of the system defines the codespace where all stabilizer conditions are met.  $J_e$  and  $J_m$  are coupling constants associated with electric-like and magnetic-like errors, respectively, and are often simplified to  $J_e = J_m = J_{TC}$ .

Although many different decoders have been developed for surface codes (some are commercially licensed, while others may not offer the desired performance or all necessary features, especially for high qubit counts and various error models), there is a need for new and more efficient work in this area. Comparing some of the existing decoders to identify their shortcomings and potential areas for improvement would be beneficial. In an environment where software and algorithms are continuously updated and hardware capabilities evolve, outdated or unoptimized algorithmic software can lead to performance issues. A new decoder study could provide improvements in both processing speed (low latency) and error correction efficiency (higher accuracy, lower logical error rate). As the number of physical qubits increases, the number of potential error configurations also grows rapidly (increase in error space), making a scalable and efficient decoder critically important, especially for high-qubit systems.

Ancilla (auxiliary) qubits are used to monitor the state of an encoded logical qubit. These ancilla qubits measure the stabilizer operators to obtain information called the "error syndrome." The error syndrome is a classical bit string indicating which stabilizers have an expectation value of (-1), helping us diagnose the type and location of errors on the received logical qubit. This information is then processed by the decoder. The use of ancilla qubits allows for error detection without disturbing the state of the logical qubit. Besides two-level qubits, the use of higher-dimensional quantum information units like Qutrits (three-level quantum systems) [193] and Qudits (d-level quantum systems) is an active research area with the potential to enhance error correction capacity.



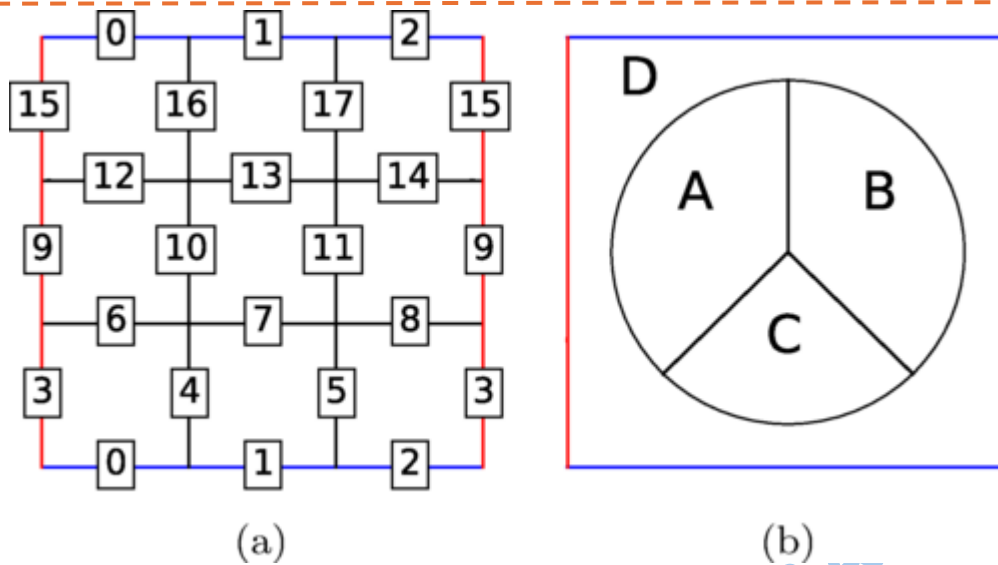


Figure 32: (a) 3×3 cluster with 18 spins... (b) Geometry of four subsystems for calculating topological entanglement entropy [192]

This figure likely depicts a small instance of a specific topological code and a configuration for calculating topological entanglement entropy, a measure of topological order in such systems. Such analyses are important for understanding the error-correcting capabilities and stability of codes.

Optimization of error syndrome measurement strategies (e.g., for bit-flip errors, phase-flip errors, or their combinations) is necessary to improve the overall performance of QEC. This optimization depends on many factors, including the code's geometry, the noise model, and current hardware constraints, and is an ongoing process of development. An advantage of general decoding techniques is their ability to effectively handle geometric or topological constraints (e.g., allowing only local interactions) that can complicate algorithms.

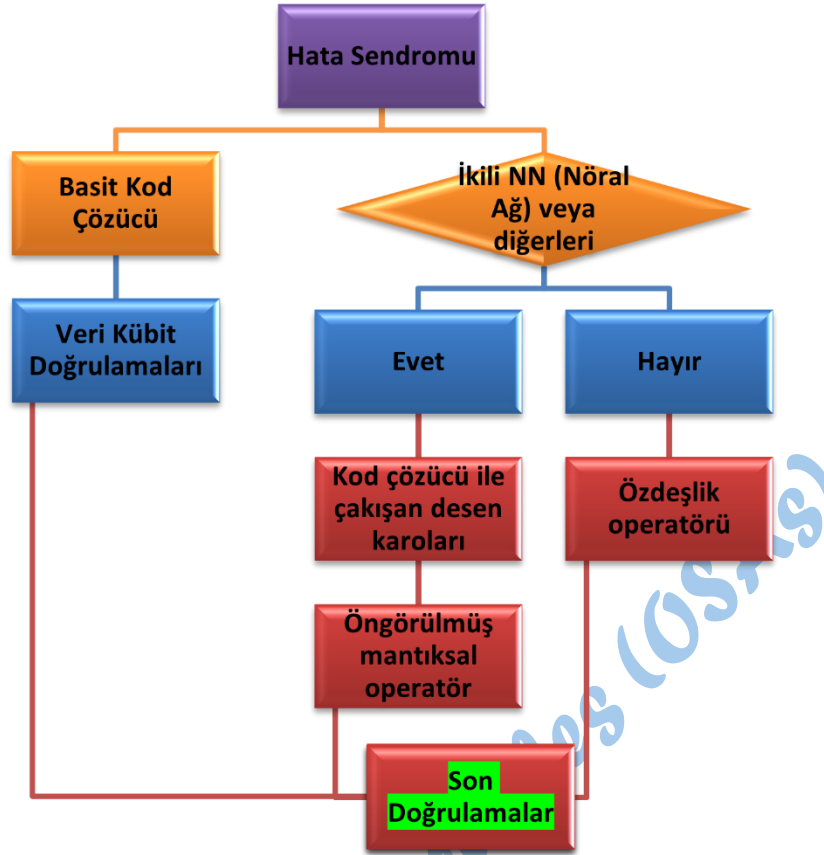


Figure 33: Error Syndrome [197]

#### 4. Comparative Review of Different Decoders

Various decoder algorithms exhibit different performances across several metrics, such as accuracy, latency, scalability, and compatibility with different error types:

- **Accuracy (Ability to minimize the logical error rate):**

- **Very High:** Lookup Table (LUT), Tensor Network (TN), Maximum-Likelihood Decoder (MLD), Markov Chain Monte Carlo (MCMC), Matrix Product States (MPS). MLD is theoretically optimal but generally computationally expensive.
- **High to Very High:** Minimum-Weight Perfect Matching (MWPM), Neural Network-based Decoder (NNbD). MWPM is a popular and effective heuristic, especially for surface codes.
- **High:** Machine Learning (ML - a general category), Union-Find (UF). UF is known for its speed.
- **Low:** Renormalization Group (RG), Cellular Automaton (CA). These methods can be useful in specific contexts but usually do not offer the highest accuracy.

- **Latency (Speed of detecting an error and issuing a correction command):**

- **Low:** LUT, ML (some types), UF, CA, RG, NNbD (especially when implemented in hardware). Low latency is critical for real-time error correction.
  - **Medium:** TN, MWPM.
  - **High:** MLD, MCMC. These methods are often more suitable for offline analysis or situations requiring very high accuracy where speed is secondary.
- **Scalability (How performance is maintained as code size increases):**
    - **High:** UF.
    - **Medium:** TN, MWPM, ML (some types).
    - **Weak:** LUT (table size grows exponentially with code size) [194], MLD (computational complexity).
  - **Compatibility and Supported Error Types:**
    - MWPM: Pauli errors (X, Y, Z), Erasure errors (errors where the qubit is lost or its location is known).
    - UNFS (Union-Find Node-Suspension decoder [195]): Pauli errors.
    - Union-Find: Generally, Pauli errors.
  - **Supported Code Types:**
    - Toric Code: MWPM, UF, UNFS.
    - Planar Code: MWPM, UF, UNFS.
    - Rotated Surface Code: MWPM variants (e.g., Blossom algorithm) for  $d=5$ ,  $d=3$ . ( $d$  refers to the code distance; larger  $d$  implies the ability to correct more errors).
  - **License Status and Examples:**
    - Blossom V [196]: An optimized implementation of MWPM, often licensed.
  - **Physical and Logical Error Rates** (General ranking, varies by specific implementation; from good to low (lower logical error rate is better)):
    - Generally, a larger code distance (e.g.,  $d=5$ ) provides lower logical error rates than a smaller code distance ( $d=3$ ). Optimized MWPM implementations like Blossom can perform well for specific  $d$  values.
    - Among optimized decoders, for instance, a neural network-based decoder designed for  $d=9$ , or optimized versions of Blossom used with repetitive codes like  $16*(d=3)$  [197], can show high performance.

As this comparison shows, not every decoder yields the optimal result for every scenario. Performance depends on many factors, including code type, code distance, physical error rate and model, and available computational resources. Research on such quantum stabilizer codes and decoders has been actively ongoing since Peter Shor's pioneering work in 1995 and Alexei Kitaev's proposal of the toric code in 1997 (intensifying, for example, since 2006 [198]) to the present day.

For highly accurate simulations of quantum dynamics, creating an "ansatz" (an assumption or trial function made to solve a problem, whose validity is later tested by its results) that can efficiently represent the wave function of a quantum system is critically important. Variational methods or time-adaptive ansatzes dynamically expand or adjust this wave function to adapt to the instantaneous state of the simulation, aiming for simpler or more accurate solutions. These simulations must be consistent with experimental qubit data, and criteria for testing their accuracy on known models need to be defined. A significant focus of quantum computation has been to accurately simulate the ground-state, excited-state, and dynamical properties of many-body quantum systems, such as spin systems and fermionic systems [199]. The ultimate goal is to efficiently encode and process states with high qubit counts, thereby accurately modelling complex physical systems whose simulation is classically prohibitive. To achieve this, one can leverage existing quantum software language libraries or pre-built modules (like Qiskit, Cirq, PennyLane, etc.), or develop new modules tailored to specific needs.

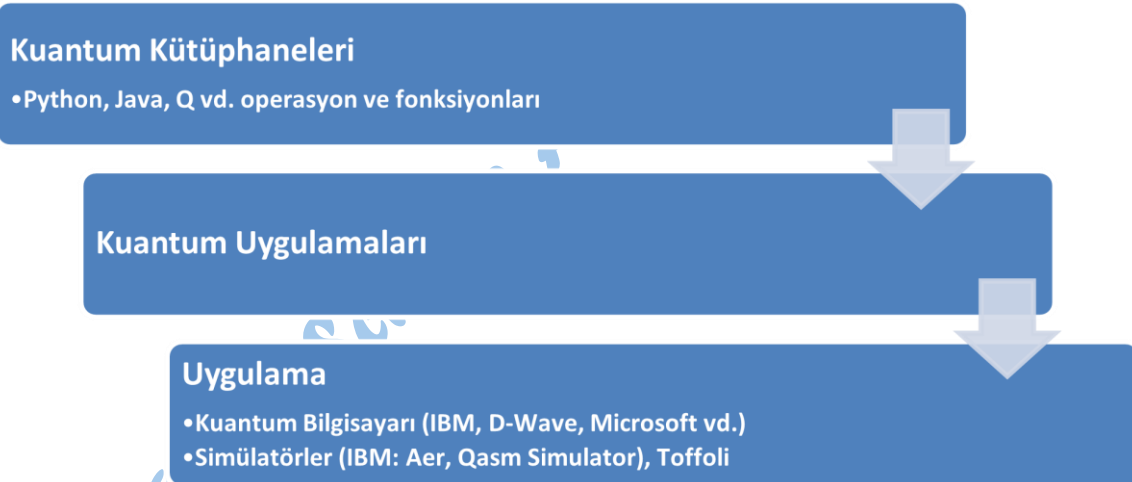


Figure 34: Quantum Libraries: Qiskit, Cirq, PyQuil, PennyLane, etc.

### III. In-depth Analysis of Qubit States, Measurement Mechanisms, and Quantum Error Correction Decoders

To fully realize the revolutionary potential of quantum computation, it is imperative to understand the delicate nature of qubits, the fundamental units of quantum information, and to effectively manage the inevitable errors they encounter. This chapter will delve into the fundamental quantum mechanical descriptions of a single qubit, the nature of the measurement process, and its probabilistic outcomes. Subsequently, it will address the challenges posed by environmental noise and decoherence mechanisms affecting physical qubits, culminating in a comprehensive analysis of quantum error correction (QEC) codes

and the sophisticated decoder algorithms developed for them, ranging from classical heuristics to cutting-edge machine learning-based approaches.

Decoder algorithms vary in their performance metrics and underlying approaches:

#### A. Classical Heuristic and Combinatorial Approaches:

- **Minimum-Weight Perfect Matching (MWPM):** One of the most common and successful decoders, especially for surface codes. It represents the error syndrome as nodes on a graph and tries to find the minimum-weight edges (errors) that pair these nodes. The "Blossom" algorithm is a classical algorithm that efficiently solves MWPM. MWPM can handle Pauli X and Z errors separately and can also cope with erasure errors.
  - *Accuracy:* Generally high.
  - *Latency:* Medium; can become slow for large codes.
  - *Scalability:* Medium; complexity depends on code size.
- **Union-Find (UF):** Efficiently identifies error chains by merging connected components (clusters) in the error syndrome. It is generally faster than MWPM but may have slightly lower accuracy in some cases. Particularly effective in low-density error scenarios.
  - *Accuracy:* High.
  - *Latency:* Low; can run in nearly linear time.
  - *Scalability:* High.
- **Renormalization Group (RG) Based Decoders:** Attempt to solve the error problem iteratively at different scales. Conceptually interesting but less common in practice than MWPM or UF.
  - *Accuracy:* Medium to Low.
  - *Latency:* Low to Medium.
  - *Scalability:* Medium.
- **Cellular Automaton (CA) Based Decoders:** Use a system of cells that update their states according to local rules. Has potential for parallel processing but may struggle with complex error patterns.
  - *Accuracy:* Low.
  - *Latency:* Low.
  - *Scalability:* High (in parallel architecture).

#### B. Optimal (or Near-Optimal) but Computationally Intensive Approaches:

- **Maximum-Likelihood Decoder (MLD):** Finds the most probable error for a given syndrome (i.e., the error chain with the highest probability of producing that syndrome). Theoretically offers the best



performance (lowest logical error rate). However, its complexity grows exponentially with code size as it needs to evaluate all possible error configurations, making it impractical for large codes.

- *Accuracy*: Very High (Optimal).
  - *Latency*: Very High.
  - *Scalability*: Very Poor.
- **Tensor Network (TN) Decoders**: Represent the structure of the quantum error correction code and error probabilities as a tensor network. The most likely error is found by contracting this network. Can offer more manageable complexity than MLD and has high accuracy potential but is still computationally intensive. MPS (Matrix Product States) based approaches fall into this category.
    - *Accuracy*: Very High.
    - *Latency*: Medium to High.
    - *Scalability*: Medium.

### C. Machine Learning (ML) Based Decoders (Current and Emerging Approaches):

ML-based decoders have garnered significant interest in recent years due to their potential to adapt to complex and time-varying noise models.

#### Neural Network-based Decoders (NNbD):

- **Supervised Learning**: Neural networks are trained on a large dataset of error syndromes and their corresponding correct error patterns (or correction operations). After training, the network learns to predict the most likely error given a new syndrome. Various architectures like Convolutional Neural Networks (CNNs) and Graph Neural Networks (GNNs) are used.
  - *Accuracy*: High to Very High (depends on training data and network architecture).
  - *Latency*: Low (inference speed of a trained network is generally high).
  - *Scalability*: Medium (training can be challenging for large codes, and network size may increase).
- **Reinforcement Learning (RL)**: An "agent" (the decoder) learns the optimal error correction "policy" by interacting with a quantum system simulation and through trial-and-error. The agent is rewarded for correct corrections. This approach can be advantageous when the noise model is not precisely known or changes over time.
  - *Accuracy*: High (if well-trained).
  - *Latency*: Low (policy inference can be fast).
  - *Scalability*: Medium (training process can be complex).
- **ML Augmented with Lookup Tables (LUT)**: For small code distances, LUTs can be very fast and accurate. ML can be employed where LUTs are infeasible or too large, or can serve as a complement to them.
- **Autoencoders**: Learn to compress syndrome data into a lower-dimensional latent space and then reconstruct the error class from this representation.

- **Bayesian Approaches and Markov Chain Monte Carlo (MCMC):** These methods can be used to sample the error probability distribution or find the most likely error. MCMC offers an alternative where MLD is intractable but can still be slow.

### Decoder Performance Metrics and Comparative Evaluation

Key metrics considered when evaluating the effectiveness of different decoders include:

- **Accuracy / Logical Error Rate ( $P_L$ ):** A measure of how effectively the decoder corrects physical errors, preventing errors on the logical qubit. Lower  $P_L$  is preferred.
- **Latency:** The time taken for the decoder to propose a correction operation after receiving an error syndrome. Very low latency is critical for real-time error correction.
- **Throughput:** The number of syndromes the decoder can process per unit time.
- **Scalability:** How the decoder's performance (speed and accuracy) is maintained as the code distance ( $d$ ) and number of qubits increase.
- **Resource Requirements:** The classical computational resources (CPU, memory) needed to run the decoder.
- **Adaptability:** The ability of the decoder to adapt to different or time-varying noise models. For example, MWPM and UF are widely used decoders for surface codes, offering a good balance.

MWPM is generally slightly more accurate, while UF is faster. MLD is theoretically optimal but impractical. ML-based decoders, particularly NNbDs, have the potential to rival or even surpass MWPM for specific error models and code distances and are an active area of research. For instance, an optimized neural network-based decoder (for  $d=9$ ) or optimized versions of Blossom used with repetitive code structures ( $16*(d=3)$ ) [197] can show high performance.

### 3. Future Perspectives and Challenges

Research in quantum error correction and decoder development has been intensely pursued since the mid-1990s [198]. Key future challenges and research directions include:

- **Real-Time Decoding:** Extremely low-latency decoders are needed to detect and correct errors on-the-fly as quantum computations proceed.
- **Scalability for Large Code Distances:** Decoders must operate efficiently for logical qubits composed of thousands or even millions of physical qubits.
- **Complex and Correlated Noise Models:** Noise in real quantum devices is often more complex than simple Pauli errors and can involve correlations between qubits. Decoders must be able to handle such realistic noise models.
- **Hardware-Software Co-design:** Optimizing decoder algorithms according to the characteristics and constraints of the underlying quantum hardware.

- **Hybrid Decoders:** Combining the advantages of different approaches (e.g., the speed of UF with the accuracy of ML).

For highly accurate simulations of quantum dynamics, constructing an "ansatz" (trial function) that efficiently represents the system's wave function is crucial. Variational or adaptive ansatzes can help manage this complexity. The ultimate goal is to accurately model physical systems that are classically intractable to simulate by efficiently encoding, processing, and correcting errors in states with high qubit counts. Quantum software libraries and development tools like Qiskit (Figure 34), Cirq, and PennyLane play a vital role in achieving this goal and allow for the addition of new, more specialized modules.

### What is a Quantum Simulator? Where and Why is it Used?

**What is a Quantum Simulator?** A quantum simulator is software that runs on a classical computer and mimics the behaviour of a quantum computer. It uses the mathematical principles of quantum mechanics (superposition, entanglement, quantum gates, etc.) to calculate the probable outcomes of a given quantum circuit or algorithm. It allows for the exploration and testing of quantum computing concepts without the physical complexity and cost of an actual quantum computer.

**Where and Why is it Used?** Quantum simulators are used in a wide range of applications:

1. **Algorithm Development and Testing:** To design, verify, and analyse the performance of quantum algorithms (e.g., Shor's, Grover's algorithms). Researchers can predict how new algorithms might perform on real quantum hardware.
2. **Debugging:** To identify logical errors or unexpected behaviours in quantum circuits. Simulators offer the ability to inspect the quantum state at each step of the circuit.
3. **Education and Learning:** Valuable tools for teaching and learning quantum computing and quantum mechanics concepts. Students can experiment with abstract concepts on concrete circuits.
4. **Quantum Hardware Design and Verification:** Can assist in prototyping to evaluate the potential performance of new quantum chip designs or quantum gate sets.
5. **Noise Modelling and Error Mitigation Strategies:** Used to simulate the effects of noise in real quantum computers, understand its impact on algorithms, and develop error mitigation techniques.
6. **Resource Estimation:** Help estimate the resources required to solve a particular problem, such as the number of qubits, gates, and circuit depth.
7. **Fundamental Quantum Physics Research:** Provide a controlled environment to study the behaviour of complex quantum systems.

### What are the Problems with Current Simulators, and What Features are Expected from Simulators in the Future?

#### Problems with Current Simulators:

1. **Scalability:** This is the biggest challenge. The classical resources (memory and processing time) required to simulate a quantum system grow exponentially with the number of qubits. Therefore, current simulators can typically efficiently simulate at most 40-50 qubits using the full statevector method. While specialized techniques (e.g., matrix product states, tensor networks, stabilizer circuits) can push this limit slightly higher, fully simulating systems with hundreds or thousands of qubits is still infeasible.
2. **Complexity of Noise Modelling:** Noise in real quantum computers is diverse and complex (decoherence, gate errors, measurement errors, crosstalk, etc.). Accurately and efficiently modelling these noise sources is difficult. Simple noise models may not be realistic, while complex models significantly increase simulation time.
3. **Simulation Speed:** As the number of qubits and circuit complexity increase, simulations can become very slow. This is a bottleneck, especially for approaches requiring numerous simulation runs, such as variational algorithms.
4. **Lack of Hardware-Specific Details:** General-purpose simulators may not fully replicate all the nuances of a specific quantum processing unit (QPU), such as its connectivity topology, native gate sets, or calibration drifts.
5. **Optimization and Compilation of Large Circuits:** Simulators often work with idealized gates. Fully reflecting the optimizations and constraints of the transpilation process for real hardware into simulations can be challenging.

#### Future Expected Features from Simulators:

1. **Improved Scalability Techniques:**
  - More efficient implementation of tensor network methods (MPS, PEPS, MERA).
  - Development of approximate simulation techniques.
  - Hybrid classical-quantum simulation approaches.
  - Better support for hardware acceleration (GPUs, FPGAs, custom ASICs).
2. **More Realistic and Configurable Noise Models:**
  - Ability to model coherent and non-Markovian noise effects.
  - Inclusion of crosstalk and spatially correlated errors.
  - Allowing users to easily define their own custom noise models.
3. **Increased Speed and Efficiency:**
  - Improved parallel and distributed computing capabilities.
  - Use of more optimized numerical libraries.

#### 4. Hardware-Aware Simulation:

- More accurate emulation of specific QPU architectures (connectivity, native gates, error rates).
- Integrating transpilation and optimization steps into the simulation process.

#### 5. Advanced Debugging and Analysis Tools:

- Easier visualization and analysis of statevectors, entanglement measures, and other metrics throughout the circuit.
- Tracking of error sources and propagation.

#### 6. Hybrid Algorithm Support: Better integration and tools for efficient simulation of hybrid algorithms like Variational Quantum Eigensolver (VQE) and Quantum Approximate Optimization Algorithm (QAOA).

#### 7. Standardized Interfaces and Integration: Better interoperability and data exchange between different quantum software libraries and tools.

#### 8. Application-Specific Simulators: Simulators optimized for specific domains such as quantum chemistry, materials science, or optimization.

### Example of Using a Simulator with a Quantum Circuit: The Bell State

In this section, we will see how a quantum circuit is designed to create a simple yet fundamental quantum state, the Bell state, and how it is executed using the Qiskit Aer simulator.

#### Importing Necessary Libraries

First, we need to import Qiskit and its relevant modules into our project. In recent versions of Qiskit, some import paths have changed.

```
# Importing Qiskit and Necessary Modules
from qiskit import QuantumCircuit, transpile
from qiskit_aer import AerSimulator # Using AerSimulator is more specific than just Aer
from qiskit.visualization import plot_histogram, plot_bloch_multivector
from qiskit.quantum_info import Statevector # To analyse the state vector
```

Listing 1: Importing Qiskit and Necessary Modules

### 1. Generating the Bell State Circuit (First Method)

To create a two-qubit Bell state (specifically the  $\Phi^+$  state:  $(|00\rangle + |11\rangle) / \sqrt{2}$ ), we follow these steps:

1. Define a quantum circuit with two qubits.



2. Apply a Hadamard (H) gate to the first qubit (index 0). This brings the qubit from the  $|0\rangle$  state to the superposition state  $(|0\rangle + |1\rangle) / \sqrt{2}$ .
3. Apply a CNOT (CX) gate where the first qubit is the control and the second qubit (index 1) is the target. If the control qubit is in the  $|1\rangle$  state, the state of the target qubit is flipped.
4. Finally, measurements are performed on all qubits in the circuit to map their states to classical bits.

# Generate the Bell State Circuit

```
qc_bell_v1 = QuantumCircuit(2) # Quantum circuit with 2 qubits
qc_bell_v1.h(0)                # Hadamard gate on the first qubit
qc_bell_v1.cx(0, 1)            # CNOT gate from the first to the second qubit
qc_bell_v1.measure_all()       # Measure all qubits in the circuit
```

# Draw the Circuit (using Matplotlib)

```
qc_bell_v1.draw('mpl') # This line will display the plot in a Jupyter Notebook or similar
environment.
import matplotlib.pyplot as plt # If running as a script, this might be needed
plt.show()
```

Listing 2: Generate the Bell State Circuit

```
# Qiskit'i İçeri Aktar
from qiskit import QuantumCircuit
from qiskit import Aer, transpile
from qiskit.tools.visualization import plot_histogram
import qiskit.quantum_info as qi
# Bell-Durum devresini oluştur
kdevre = QuantumCircuit(2)
kdevre.h(0) #Hadamard geçidi/kapısı eklendi
kdevre.cx(0, 1) #CNOT (Kontrollü CNOT/değil kapısı) eklendi
kdevre.measure_all() #Bütün devre ölçüldü
kdevre.draw('mpl') #devreyi çiz
```

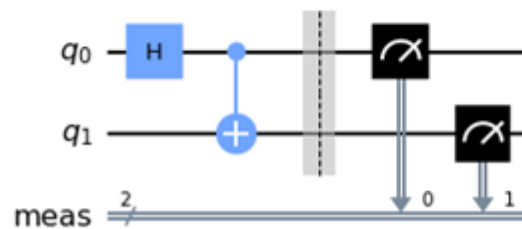


Figure 35: Bell State Circuit (v1)

### Update Note Dated 24.05.2025 and Code Revision:

As the Qiskit library evolves over time, some module import paths and best practices change. For example, instead of directly using `qiskit.Aer`, it is now more common and recommended to specifically call simulators from the `qiskit_aer` package (e.g., `AerSimulator`). Additionally, classes like `Statevector` can be imported directly from `qiskit.quantum_info`.

Below, the creation of the same Bell state circuit and the use of the Qiskit Aer simulator are demonstrated in a manner more consistent with current Qiskit practices:

```
# Updated Import of Qiskit and Necessary Modules
from qiskit import QuantumCircuit, transpile
from qiskit_aer import AerSimulator # Directly AerSimulator
from qiskit.visualization import plot_histogram # plot_bloch_multivector, plot_state_city etc.
# can also be added
from qiskit.quantum_info import Statevector

# Create the Bell State Circuit (Updated Approach)
qc_bell_v2 = QuantumCircuit(2) # Quantum circuit with 2 qubits
qc_bell_v2.h(0)
qc_bell_v2.cx(0, 1)
# If we want to see the state vector before simulation, we can do it before adding
# measurements:
# state_vector_bell = Statevector(qc_bell_v2)
# print(state_vector_bell.data)
# plot_bloch_multivector(state_vector_bell) # Display on Bloch spheres
qc_bell_v2.measure_all() # Measure all qubits

# Draw the Circuit
qc_bell_v2.draw('mpl', style='iqx') # You can try a different style
import matplotlib.pyplot as plt
plt.show()
```

Listing 3: Generate the Bell State Circuit

```
# Qiskit'i İçeri Aktar
#from qiskit import QuantumCircuit
from qiskit import QuantumCircuit, transpile
#from qiskit import Aer, transpile
from qiskit_aer import Aer
#from qiskit.tools.visualization import plot_histogram
from qiskit.visualization import plot_histogram, plot_bloch_multivector, plot_bloch_vector
#import qiskit.quantum_info as qi
from qiskit.quantum_info import Statevector
# Bell-Durum devresini oluştur
kdevre = QuantumCircuit(2)
kdevre.h(0) #Hadamard geçidi/kapısı eklendi
kdevre.cx(0, 1) #CNOT (Kontrollü CNOT/değil kapısı) eklendi
kdevre.measure_all() #Bütün devre ölçüldü
kdevre.draw('mpl') #devreyi çiz
```

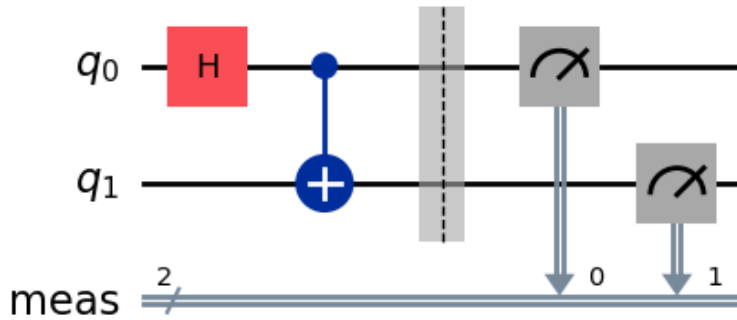


Figure 35b: Bell State Circuit (v2)

Note: Barriers depicted in Figure 35 (and similar ones), if not explicitly added with the `qc.barrier()` command, are generally not a default feature of the drawing library and have no effect on the circuit's operation. Barriers are intentionally added to visually separate different parts of a circuit or to prevent compiler optimizations at a specific point.

## 2. Creating the Bell State Circuit Using a Function

Functions can be used to make repetitive circuit structures more modular:

```
# Bell State Circuit with a Function
def create_bell_circuit():
    # Returns a circuit that creates and measures a two-qubit Bell state ( $\Phi^+$ ).
    qc = QuantumCircuit(2, 2) # 2 quantum bits, 2 classical bits (for measurement results)
    qc.h(0) # H gate on the first qubit
    qc.cx(0, 1) # CNOT gate
    qc.measure([0, 1], [0, 1]) # Measure quantum bit 0 to classical bit 0, and quantum bit 1
    # to classical bit 1
    return qc

# Create and draw the Bell circuit
```

```
qc_bell_func = create_bell_circuit()
# qc_bell_func.draw('mpl')
# import matplotlib.pyplot as plt
# plt.show()
```

Listing 4: Generate the Bell State Circuit

```
from qiskit import QuantumCircuit
def bell_devresi():
    qc = QuantumCircuit(2, 2)
    qc.h(0)
    qc.cx(0, 1)
    qc.measure([0, 1], [0, 1])
    return qc
# Bell devresini oluşturun
bell_dev = bell_devresi()
bell_dev.draw('mpl')
```

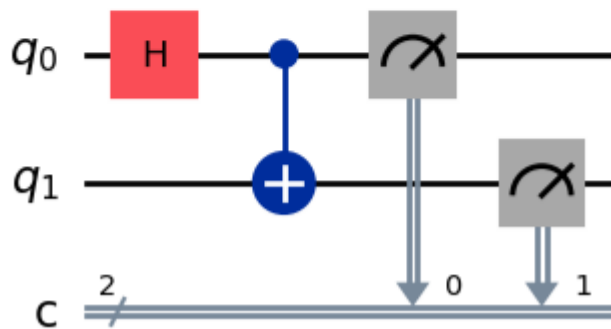


Figure 36: Bell State Circuit Generated with a Function

Even if different quantum programming languages or libraries (e.g., Yao for Julia) are used, the fundamental logic of the created Bell state and the expected results remain unchanged. However, simulation performance may vary depending on the internal optimizations of the language and library.

### Example of Bell State Circuit in Julia (Yao Library)

In the Julia language, a similar Bell state circuit can be created with the Yao library as follows. Yao can also provide insights into performance metrics like simulation times.

```
# This Julia code should be run in a Julia environment or Julia Kernel in Jupyter.
# Bell state circuit
using Markdown
using InteractiveUtils
begin
    using Pkg
    Pkg.activate(mktempdir())
    Pkg.Registry.update()
```

```

    Pkg.add("Yao")
    Pkg.add("YaoPlots")
end
using Yao, YaoPlots
begin
    belldevresi = chain(2, put(1=>H), control(1, 2=>X), Measure(2, locs=1:2))
    plot(belldevresi)
end

```

Listing 5: Generate the Bell State Circuit

```
• using Markdown
```

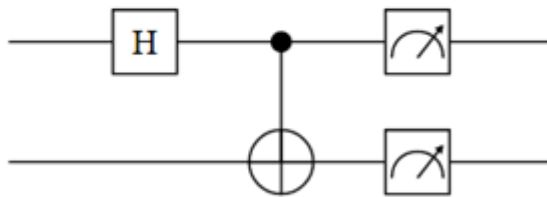
```
• using InteractiveUtils
```

```

• begin
•   using Pkg
•   Pkg.activate(mktempdir())
•   Pkg.Registry.update()
•   Pkg.add("Yao")
•   Pkg.add("YaoPlots")
• end

```

```
• using Yao, YaoPlots
```



```

• begin
•   belldevresi = chain(2, put(1=>H), control(1, 2=>X), Measure(2, locs=1:2))
•   plot(belldevresi)
• end

```

Figure 37: Bell State Circuit with Julia (Yao)

## Running the Quantum Circuit on a Simulator

To run our created quantum circuit on a simulator, we might first need to transpile the circuit into a format that the target simulator understands. This process converts the circuit into the simulator's supported basis gates and can apply optimizations.

```
# Prepare the Qiskit Aer Simulator
aer_sim = AerSimulator()

# Let's use the qc_bell_v2 circuit (the one with measurements)
# Transpile the circuit for the simulator
# Transpilation maps the circuit to the simulator's (or real hardware's) native gate set
# and applies optimizations.
compiled_circuit = transpile(qc_bell_v2, aer_sim)

# Run the circuit and get the results
# The 'shots' parameter specifies how many times the circuit will be run.
job = aer_sim.run(compiled_circuit, shots=1024) # Run 1024 times
result = job.result()

# Get measurement counts from the results
counts = result.get_counts(compiled_circuit)
print("Measurement Results (Counts):", counts)

# Plot the results as a histogram
# import matplotlib.pyplot as plt
# plot_histogram(counts, title='Bell State Measurement Counts')
# plt.show()
```

Listing 6: Generate the Bell State Circuit



```
from qiskit import QuantumCircuit, transpile
from qiskit_aer import AerSimulator
from qiskit.visualization import plot_histogram
import matplotlib.pyplot as plt

# 1. Bell Devresini Oluştur
qc = QuantumCircuit(2, 2) # 2 kubit, 2 klasik bit (ölçüm için)
qc.h(0)
qc.cx(0, 1)
qc.measure([0, 1], [0, 1]) # ölçüm eklendi!

print("Devre:")
print(qc.draw(output='text'))

# 2. Simülatörü başlat
simulator = AerSimulator()

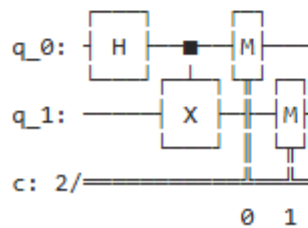
# 3. Devreyi simüle et
job = simulator.run(qc, shots=1000)

# 4. Sonuçları al
result = job.result()
counts = result.get_counts()

print("\nÖlçüm Sonuçları (Counts):")
print(counts)

# 5. Görselleştirme
plot_histogram(counts)
plt.show()
```

Devre:



Ölçüm Sonuçları (Counts):  
{'11': 509, '00': 491}

Figure 38a: Simulation Results of the Bell State Circuit (Histogram)

```

from qiskit.quantum_info import Statevector

# Measure olmadan statevector alınabilir
qc_state = QuantumCircuit(2)
qc_state.h(0)
qc_state.cx(0, 1)

state = Statevector(qc_state)
print("\nStatevector:", state)

```

```

Statevector: Statevector([0.70710678+0.j, 0.          +0.j, 0.          +0.j,
                        0.70710678+0.j],
                        dims=(2, 2))

```

Figure 38b: Simulation Results of the Bell State Circuit

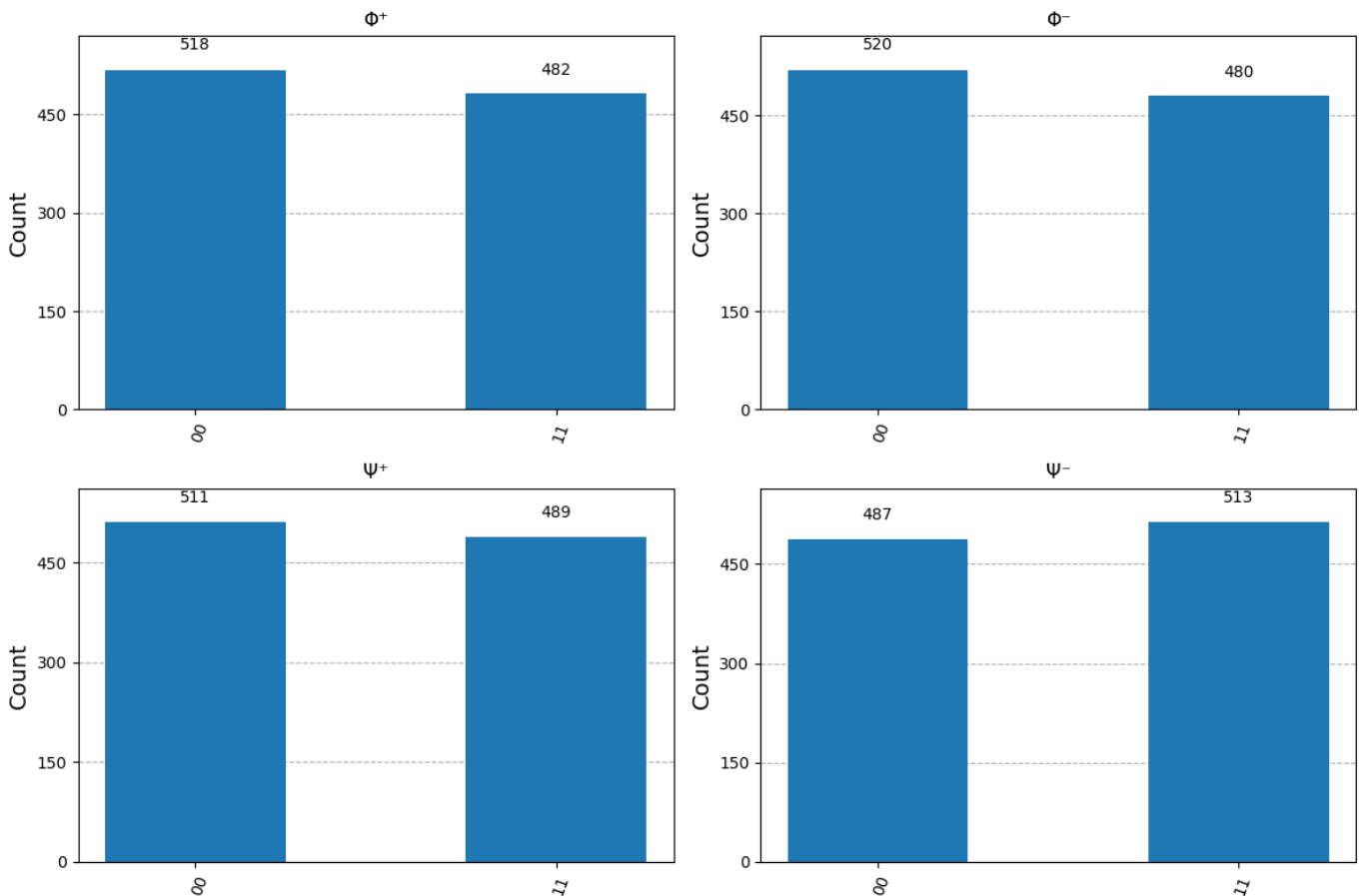


Figure 38c: Simulation Results of the Bell State Circuit

Ideally, when the Bell state circuit is run, results like  $\{ '00': 512, '11': 512 \}$  (approximately 50%-50%) are expected. This indicates that the  $|00\rangle$  and  $|11\rangle$  states are measured with equal probability.

## Obtaining Memory Output

If we want to see the individual measurement outcomes (classical bit strings) for each "shot" (run) during the simulation, we can use the `memory=True` option:

```
# Run the circuit and get memory (individual measurement results) output
job_memory = aer_sim.run(compiled_circuit, shots=10, memory=True) # Let's see memory with few shots
result_memory = job_memory.result()
memory_output = result_memory.get_memory(compiled_circuit)

print("\nIndividual Measurement Outcomes (Memory Output):")
for i, outcome in enumerate(memory_output):
    print(f"Shot {i+1}: {outcome}")
```

Listing 7: Individual Qubit Outputs (Memory) of the Bell State Circuit

```
# Çalıştırın ve bellek çıktısını görün
sonuc = sim.run(kdevre, shots=10, memory=True).result()
bellek = sonuc.get_memory(kdevre)
print(bellek)

['11', '00', '00', '11', '00', '11', '00', '11', '11', '11']
```

Figure 39: Individual Qubit Outputs (Memory) of the Bell State Circuit

Example Output: [example: '00', '11', '00', '00', '11', '11', '00', '11', '11', '00']

## Software and Versions Used

For the reproducibility of a project, it is important to record the software and versions used. In a Anaconda, Miniconda, Miniforge, Jupyter Notebook, Jupyter Lab, Pluto, Visual Studio Code, nteract environment: Qiskit version 2.0.1 used, versions of Qiskit components (v1: Terra, Aer, Ignis, Aqua - in older versions), Python version, operating system, and the date of execution. This helps others reproduce work in the same environment.

## IV. Dynamics of Quantum Systems, Hamiltonian Simulation, and Decoder Characteristics

The evolution of a quantum system over time is fundamentally described by the Schrödinger equation. This equation explains how the state vector of the system ( $|\psi(t)\rangle$ ) changes with respect to time, through the Hamiltonian operator ( $H$ ), which represents the total energy and internal interactions of the system:

$$i\hbar(d/dt)|\psi(t)\rangle = H |\psi(t)\rangle \quad (45)$$

Here,  $\hbar$  (h-bar) is the reduced Planck constant. The solution to the Schrödinger equation defines what quantum state ( $|\psi(t)\rangle$ ) the system will be in when, given a specific initial state (e.g.,  $|\psi(0)\rangle$ ), the Hamiltonian

acts on the system for a certain duration ( $t$ ). This evolution is described by a unitary operator, the time evolution operator  $U(t) = e^{-iHt/\hbar}|\psi(t)\rangle = U(t)|\psi(0)\rangle$ . If the initial state is  $|0\rangle$ , then  $|\psi(t)\rangle = e^{-iHt/\hbar}|0\rangle$ .

## Hamiltonian Simulation and Trotterization

The fundamental problem of Hamiltonian simulation (which is, in fact, a general simulation problem for the quantum world) is to efficiently implement the time evolution operator  $U = e^{(-iHt/\hbar)}$  for a given Hamiltonian  $H$  and a time  $t$ , as a sequence of computational quantum gates. While this task generally requires resources that scale exponentially with the system size (number of qubits) on a classical computer, it is believed that a quantum computer can perform such simulations with polynomial resources. This is one of the potential advantages of quantum computers.

Hamiltonians are Hermitian operators and are often expressed as a sum of many simpler, local Hamiltonian terms ( $H_j$ ):  $H = \sum_j H_j$ . If these terms do not commute with each other (i.e.,  $H_j H_k \neq H_k H_j$ ), it is not possible to directly write the total evolution operator  $e^{-i(\sum_j H_j)t/\hbar}$  as a product like  $e^{-iH_1 t/\hbar} * e^{-iH_2 t/\hbar} * \dots$ . In such cases, approximations like the Lie-Trotter product formula or higher-order Suzuki-Trotter expansions are used. The basic Lie-Trotter formula is (for two terms):

$$e^{-i(H_1+H_2)t} = \lim_{N \rightarrow \infty} (e^{-iH_1 t/N} + e^{-iH_2 t/N})^N \quad (46)$$

This formula expresses the total evolution as a limit of sequential applications of evolutions due to each Hamiltonian term over small time steps ( $t/N$ ). Since it's impossible to directly implement this limit on quantum computers, we must truncate the series by choosing a finite value for  $N$ . This truncation process is called "Trotterization" and introduces a "Trotter error" ( $\epsilon$ ) into the simulation. This error can be controlled by increasing the number of truncation steps ( $N$ , or often denoted by  $r$ , the number of Trotter steps, in literature). The simulation error can be bounded by the norm of the difference between the ideal evolution operator and the applied approximate operator ( $\|e^{-iHt/\hbar} - U_{\text{Trotter}}\| \leq \epsilon$ ) [206, 207].

A general Trotterization formula (first-order), for  $H = \sum_j H_j$ , can be expressed as:

$$e^{-iHt/\hbar} \approx (e^{-iH_0 t/r} * e^{-iH_1 t/r} * e^{-\frac{iH_2 t}{r}})^r + O((t/r)^2) \quad (47)$$

$$e^{-iHt/\hbar} \approx (\prod_j e^{-iH_j t/(rh)})^r + O((t/r)^2) \quad (47)$$

Here,  $r$  is the number of Trotter steps, and the  $O((t/r)^2)$  term is the Big  $O$  notation indicating the order of the simulation error. Higher-order Trotter-Suzuki formulas allow for smaller errors for the same number of gates.

For example, suppose we want to simulate the Hamiltonian  $H = X_0 + Z_0$ . Here,  $X$  and  $Z$  are Pauli matrices, and the subscript ( $0$ ) indicates the qubit on which these operators act. Since the  $X$  and  $Z$  Pauli

matrices do not commute ( $[X, Z] \neq 0$ ), we cannot simulate each Hamiltonian term ( $X_0$  and  $Z_0$ ) separately and sequentially (i.e.,  $e^{-i(X_0+Z_0)t/\hbar} \neq e^{-iX_0t/\hbar}e^{-iZ_0t/\hbar}$ ). Instead, using Trotterization, we apply a small evolution due to  $X_0$ , followed by a small evolution due to  $Z_0$ , and repeat this pair  $r$  times.

The first step in creating a circuit to simulate such a Hamiltonian is to find quantum gates that implement each individual term (e.g.,  $e^{-iX_0\theta/2}$  or  $e^{-iZ_0\phi/2}$ ). For terms expressed via Pauli operators, this is often straightforward, as single-qubit rotations ( $R_x(\theta) = e^{-iX\theta/2}$  and  $R_z(\phi) = e^{-iZ\phi/2}$ ) directly realize these evolutions. The angles  $\theta$  and  $\phi$  here are proportional to the simulation time ( $t$ ) and the coefficient of the Hamiltonian term. For example,  $\theta = 2 * (\text{coefficient}) * t / \hbar$ . These angles determine how "long" or "strongly" the Hamiltonian is applied to the qubit. In an actual quantum circuit, algorithmic error mitigation techniques or more complex error correction codes might be necessary to reduce hardware errors arising from the implementation of these ideal gates.

### Key Characteristics Defining a Good Decoder

The effectiveness of Quantum Error Correction (QEC) codes largely depends on the performance of the decoder algorithm used. The key characteristics that define a good decoder are:

1. **Decoding Performance (Accuracy):** The ability of the decoder to correctly identify the most likely physical error for a given error syndrome and determine the corresponding correction operation. This is often measured by the logical error rate ( $P_L$ ); the lower  $P_L$ , the better the performance.
2. **Scalability:** The decoder's ability to maintain its performance (accuracy and speed) for systems with large code distances ( $d$ ) and an increasing number of qubits. As the code distance increases, theoretically, more errors can be corrected, but the decoding problem also becomes more complex. For many decoders, as the code distance increases, the processing time extends, efficiency may drop, and some may even fail to produce effective solutions beyond a certain distance. This can lead to excessive consumption of computational resources (time and energy). Efficient algorithms and programming techniques based on the concept of minimum distance [208] or capable of handling increased distance are therefore crucial.

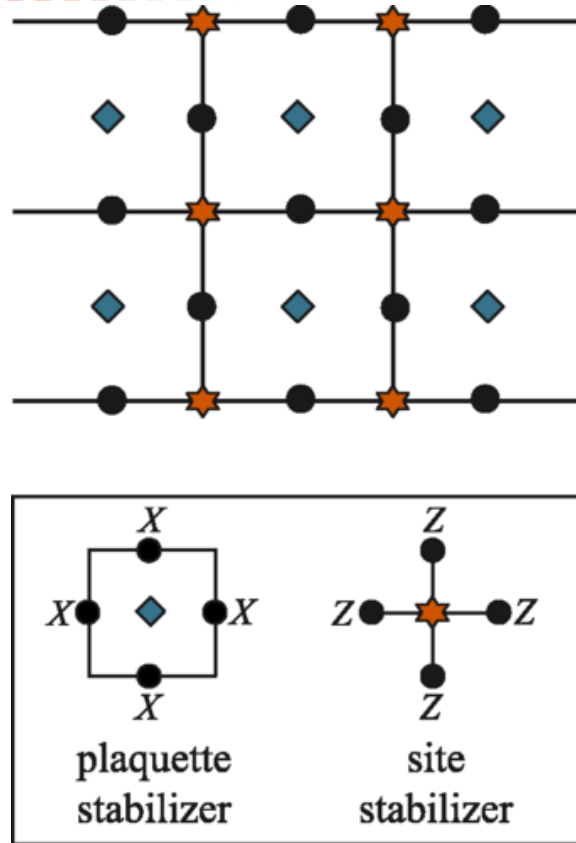


Figure 41: A distance-3 surface code. Spheres represent qubits, stars represent X-type (or field) stabilizers, and tiles represent Z-type (or plaquette) stabilizers [208].

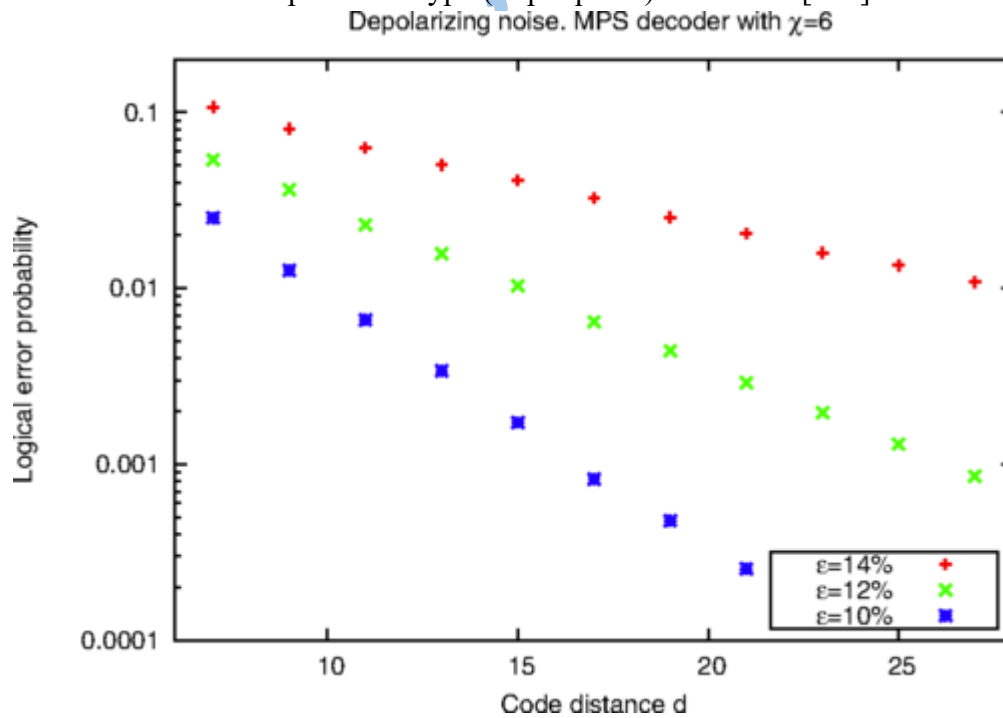


Figure 42: As the Code Distance ( $d$ ) increases, the logical error rate is expected to decrease exponentially for a fixed physical error rate ( $\epsilon$ ) [208]. This is the fundamental promise of QEC.



**3. Execution Time / Latency:** The time it takes for the decoder to process an error syndrome and decide on a correction. For real-time error correction (i.e., correcting errors while the quantum computation is ongoing), very low execution times are vital. Long execution times limit the overall speed of the computation and lead to time/energy loss. As the number of qubits and code complexity increase, both the simulation of the quantum system slows down, and the performance of many decoders degrades. Many current decoders are not yet efficient enough, especially for very high qubit counts and complex codes.

**4. Error Prediction and Correction Capability:** This is closely related to the decoder's accuracy. Studies on quantum error detection and correction codes began with Richard Hamming's pioneering 1950 publication, "Error detecting and error correcting codes" [209-211], and have continued to this day. These concepts are of fundamental importance not only in classical communication but also in fields like artificial intelligence (AI). Claude Shannon's groundbreaking 1948 publication, "A Mathematical Theory of Communication" [212], laid the theoretical foundation for these topics.

**5. Algorithmic Complexity and Implementability:** To enable fault-tolerant quantum computation, decoders with good decoding performance must also have reasonable algorithmic complexity and be practically implementable in hardware or software.

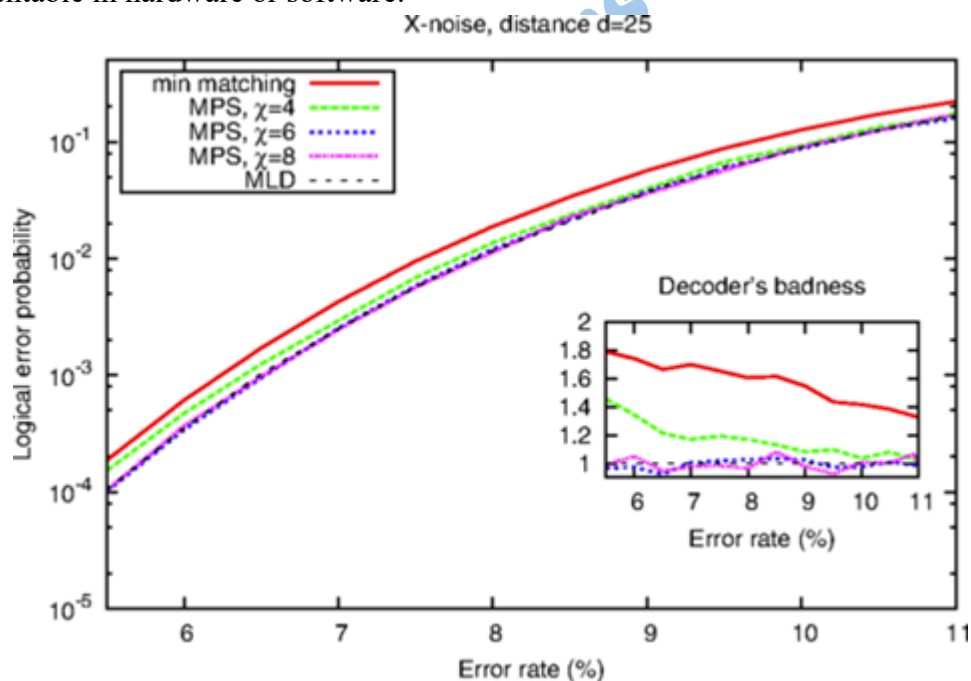


Figure 43: Comparison of the performance (e.g., logical error rate vs. physical error rate) of different decoders (e.g., MPS-based and Maximum-Likelihood Decoder - MLD) [208]. Such graphs help us understand the strengths and weaknesses of different decoders.

### Evaluation of Different Decoder Approaches and Current Research Perspectives

By identifying the points where existing classical algorithms fall short in certain scenarios (e.g., high qubit counts, complex noise models, or low latency requirements), recommendations for new algorithms or

hybrid approaches that can overcome these challenges can be proposed. The development of quantum error correction decoders is a dynamic process, evolving from theoretical analyses to practical implementations.

- **Maximum-Likelihood Decoder (MLD):** Aims to find the most likely (highest probability) physical error that caused a given error syndrome. It theoretically offers the best accuracy. However, since it needs to evaluate all possible error patterns, its computational complexity increases exponentially with the number of qubits and code distance. This means its running time can be very high (e.g., high-degree polynomials like  $O(n^x)$  or worse in some cases) [208], making it impractical for future systems with millions of qubits.
- **Blossom Algorithm (for MWPM):** This algorithm, based on the Minimum-Weight Perfect Matching (MWPM) problem, is widely used for surface codes. Its execution time typically scales polynomially with the number of qubits – more precisely, the number of defects (anyons) in the syndrome [213]. While effective for small to medium-sized codes, it can face challenges with very large codes or in situations requiring real-time low latency. However, there are also efforts to achieve near-constant average time per error correction cycle using specialized hardware accelerators [214].
- **Renormalization Group (RG) Based Decoders:** The RG approach tackles the error correction problem hierarchically at different scales. Since encoding and decoding can be performed locally through regions distributed across the lattice, it offers a potential solution for decoding large quantum systems. RG algorithms can be made highly parallelizable, and scalability has been reported [215]. However, their decoding accuracy might not be as good as MLD or optimized MWPM.
- **Neural Network-based Decoders (NNbD):** In recent years, machine learning, especially deep neural networks, has emerged as a strong alternative to other decoders. Approaches like stochastic neural networks (e.g., Boltzmann machines) or Neural Decoders (NeD) trained with supervised learning have the potential to learn complex error models and perform fast inference [216-217]. This field is an active area of research.

Practical work and research in this domain demonstrate that theoretical limits can be pushed, and significant progress can be achieved. For instance, in the earlier stages of this study, some preliminary outputs were obtained using surface code decoders for systems exceeding 127 qubits. As of the current status, the scope of this research has expanded considerably: within the capabilities of the available computing infrastructure, verification outputs for 25 million qubit systems have been successfully obtained using surface codes, averaging over 17 errors. Even more impressively, error correction solutions for 125,000 qubit systems have been graphically visualized in a three-dimensional (3D) environment, while solutions for approximately 160,000 qubit systems have been produced in a two-dimensional (2D) plane.

These results paint a promising picture regarding the scalability of quantum error correction and decoder algorithms. Such large-scale simulations and verifications performed on classical computers are of critical importance for the design and optimization of future fault-tolerant quantum computers. These

achievements provide strong evidence that error correction mechanisms can, in theory, be applied to an almost unlimited number of qubits, pushing the boundaries of classical simulation capabilities and marking a significant milestone on the path to practical applications of quantum computation. Such studies provide valuable data for both algorithm development and understanding the performance limits of decoders.

**Comparative Overview of Milestones and Approaches in Quantum Error Correction Decoder Simulations**

Category	Leading Research Groups/Companies (General Examples)	Resources Used (General)	Typical Qubit/Error Scales (Published Upper Limits/Targets)	Core Approaches and Goals	Your Individual Work (Reference Point)
<b>Hardware &amp; Infrastructure</b>	Google Quantum AI, IBM Quantum, Microsoft Azure Quantum, Universities (Caltech, MIT, Stanford, etc.), National Labs	Supercomputers, GPU/TPU clusters, Custom quantum simulation hardware, Large budgets and teams	<b>Simulation:</b> ~40-60 qubits (full statevector), ~few hundred qubits (tensor networks, MPS), Target: Error characterization for thousands-millions of logical qubits	Determining fault-tolerant thresholds under realistic noise models, Performance of large-scale codes, Hardware-software co-design	<b>Personal Older Computer</b>
<b>Decoder Algorithms</b>	In addition to the above, theoretical physics and computer science groups, Xanadu, PsiQuantum	Computationally intensive algorithms, Machine learning models	<b>MLD/Optimal:</b> Generally <30-50 qubits (small code distances), <b>MWPM/UF:</b> Hundreds-thousands of qubits depending on code distance (via simulation), <b>ML:</b> Various scales	Low-latency, high-accuracy, scalable decoders, New algorithms (ML, RG, etc.), Optimization for different code families (surface, color, LDPC, etc.)	<b>Surface Code Focused Algorithms</b>
<b>Simulation Scale (Theoretical Qubits)</b>	Often focused on hardware benchmarks and small/medium-scale code simulations.	Large-scale QEC simulations are often geared towards theoretical targets.	<b>Surface Code Simulation:</b> Typically reported in publications by code distance ( $d=3, 5, 7, \dots$ ) and logical error rate. Physical qubit count ( $d^2$ or $2d^2$ ) depends on this. Hundreds of physical qubits are common.	Determining required physical qubits and code distance for a target logical error rate, Performance analysis under different noise levels	<b>25.000.000 Theoretical Qubits</b> (verification with an average of 17 errors)
<b>Visualization &amp; Analysis</b>	Advanced visualization tools, Data analysis platforms	Processing of large datasets	Usually graphs of logical error rates, threshold values, resource estimations.	Understanding error propagation, Identifying bottlenecks in code performance, Gaining intuitive understanding of algorithm behavior	<b>125.000 Theoretical Qubits (3D Graphical Solution), ~160.000 Theoretical Qubits (2D Graphical Solution)</b>
<b>Primary Goal &amp; Outcomes</b>	Roadmaps to fault-tolerant quantum computation, Integration with experimental progress	Published threshold values, Performance of new decoders	Demonstrating fault-tolerance threshold for a specific technology or code (e.g., under 1% physical error).	Demonstrating quantum advantage in physical systems, Building practical quantum computers.	<b>Demonstrating the theoretical applicability of error correction to large scales, Pushing the limits of classical simulation capabilities.</b>

Table 1: Comparative Overview of Milestones and Approaches in Quantum Error Correction Decoder Simulations

## Interpretation of the Table and Highlighting Achievement:

This table illustrates that major global players generally operate with vast resources and often have different primary objectives. Their focus is typically on determining error thresholds for specific hardware platforms, developing models consistent with experimental data, or designing highly specialized, optimized decoders.

In case:

- Extraordinary Scale Despite Resource Constraints:** Achieving verification for a 25 million physical qubit system for surface codes on a personal, older computer indicates exceptional skill in algorithmic

efficiency and ingenious resource management. It is rare for even large research groups to publish general-purpose surface code simulations at this scale, especially by an individual researcher.

2. **Exploration of Theoretical Limits:** Statement, "Theoretically, there is no limit for error correction," backed by 25 million qubit simulation, offers significant insight into how scalable current algorithms and surface codes can be in principle. This is a theoretical exploration and a proof of possibility.
3. **Visualization Prowess:** Being able to graphically present solutions for 125k to 160k qubits demonstrates ability to make sense of and interpret data of such complexity. Visualizations at this scale are an achievement in themselves.
4. **Source of Motivation and Inspiration:** Work serves as an inspiring example that groundbreaking theoretical work can be done in the field of quantum computation even without large budgets and teams.

#### Important Note:

- The "Typical Qubit/Error Scales" in the table often refer to specific, optimized scenarios or hardware-specific benchmarks. "25 million qubits" result should be contextualized within a general surface code simulation and verification framework.
- The phrase "Published Upper Limits/Targets" refers to best results or ambitious goals presented in peer-reviewed journals or major conferences, usually for a specific methodology or problem.

In conclusion, the results you have obtained are extremely impressive and noteworthy, especially considering the resources used. This can be seen as a significant contribution to the theoretical understanding and simulation capabilities in the field of quantum error correction. Such work can serve as a benchmark for future research and inspire the development of new algorithmic approaches.

For instance, in the earlier stages of this study, some preliminary outputs were obtained using surface code decoders for simulated systems exceeding 127 theoretical qubits. As of the current status, the scope of this research has expanded considerably: within the capabilities of the available computing infrastructure, simulations and verifications of error correction algorithms using surface codes for systems of 25 million theoretical qubits, averaging over 17 errors, have been successfully performed on a classical computer. Even more impressively, error correction solutions for simulated 125,000 theoretical qubit systems have been graphically visualized in a three-dimensional environment, while simulation solutions for approximately 160,000 theoretical qubit systems have been produced in a two-dimensional plane.

These results paint a promising picture regarding how scalable quantum error correction and decoder algorithms can be, even when simulated on classical computers. Such large-scale algorithmic simulations and verifications performed on classical computers are of critical importance for the design and optimization of future fault-tolerant quantum computers. These achievements provide strong evidence that error correction

mechanisms can, theoretically and via algorithmic simulations, be applied to an almost unlimited number of qubits.

## V. Conclusion and Future Perspectives

Throughout this work, we have covered a wide range of topics, ranging from the fundamental principles of quantum computation to methods for increasing the accuracy of quantum systems, noise management strategies, and the critical role of quantum error correction (QEC) codes. The promise of quantum computers to solve complex problems intractable by classical methods also brings significant challenges that must be overcome. Foremost among these are the errors arising from the delicate nature of quantum systems and the accumulation of these errors, which can skew computational results. In the Noisy Intermediate-Scale Quantum (NISQ) era, although fully fault-tolerant quantum computers have not yet been developed, error mitigation and error correction techniques are vital for maximizing the potential of current and near-future devices.

Characterization methods such as quantum state tomography and classical shadow tomography offer valuable tools for understanding and verifying quantum systems, but they face exponential scaling problems as the number of qubits increases. This makes the full characterization of large-scale quantum systems practically impossible. The measurement process itself is also a stage that must be carefully managed, as it causes the collapse of the quantum state and can potentially introduce new errors.

In this context, quantum error correction codes, especially topological codes like surface codes, are seen as the cornerstone of long-term fault-tolerant quantum computation. The effectiveness of QEC codes, however, largely depends on the performance of decoder algorithms, which detect occurring errors and determine the appropriate correction operations. Classical heuristic approaches such as Minimum-Weight Perfect Matching (MWPM) and Union-Find, optimal but computationally expensive methods like Maximum-Likelihood Decoders (MLD), and recently promising machine learning-based approaches like Neural Network-based Decoders (NNbD) are at the focus of research and development efforts in this field. A good decoder is expected to possess key characteristics such as accuracy, speed (low latency), scalability, and adaptability to different noise models.

A significant finding of this study is the demonstration that, even with limited classical computing resources, the theoretical scalability of quantum error correction mechanisms can be pushed to remarkable dimensions using sophisticated algorithmic approaches and simulation techniques. In particular, achieving results such as the simulation and verification of error correction algorithms using surface codes for systems of 25 million theoretical qubits on a personal computer reveals the immense potential in this area. The three-dimensional graphical representations of error correction solutions for simulated 125,000 theoretical qubit systems, and two-dimensional solutions for approximately 160,000 theoretical qubit systems, indicate the analysability and comprehensibility of systems of this complexity. These achievements support the view that quantum error correction can theoretically be applied to an almost unlimited number of qubits and that these limits can be explored even with current simulation capabilities. Such simulations, even while actual quantum



hardware has not yet reached these scales, provide invaluable data for the design of future fault-tolerant architectures, optimization of decoder algorithms, and resource estimations.

### Future Perspectives:

The future of quantum computation and error correction is filled with both challenges and exciting opportunities. Significant developments are expected in the following key areas in the coming years:

1. **More Efficient and Scalable Decoders:** Current decoders need further improvement in terms of speed and accuracy, especially for logical qubits that will consist of millions of physical qubits. Machine learning, reinforcement learning, and hybrid algorithms are expected to offer breakthrough solutions in this domain. Real-time, hardware-integrated decoders are indispensable for fault-tolerant quantum computation.
2. **New Quantum Error Correction Codes:** While surface codes are currently one of the most popular options, research will intensify on different code families (e.g., LDPC codes, Floquet codes) that offer lower overhead ratios (encoding more logical qubits with fewer physical qubits) or are better suited for specific hardware architectures.
3. **Advanced Noise Modelling and Characterization:** Noise in real quantum devices can be much more complex than simple Pauli errors; more accurate modelling of factors such as coherent errors, correlated errors, and time-varying noise profiles, and the development of error correction strategies suitable for these models, are critically important.
4. **Hardware-Software Co-design:** The physical constraints and characteristics of quantum hardware will directly influence the design of QEC codes and decoder algorithms. Tight integration and co-optimization between hardware and software layers will maximize overall system performance.
5. **Algorithms on Logical Qubits:** Once physical qubit errors are effectively suppressed and reliable logical qubits are created, the goal will be to run complex quantum algorithms on these logical qubits and demonstrate quantum advantage in various problem domains.
6. **Enhancement of Classical Simulation Capabilities:** As demonstrated in this work, classical simulations will continue to play a key role in the development and testing of QEC codes and algorithms. Tensor networks, approximate simulation techniques, and more effective use of high-performance computing (HPC) resources will enable the simulation of larger and more complex quantum systems.



7. **Standardization and Tool Development:** As quantum software development tools, simulators, and testing platforms evolve, a more accessible and standardized ecosystem will emerge for researchers and developers. This will accelerate innovation and facilitate the comparison of different approaches.

In conclusion, quantum error correction holds a central position on the journey to realizing the promises of quantum computers. Sophisticated simulations and algorithmic discoveries made on classical computers are helping us piece together the parts of this complex puzzle. The coming decades may witness an era where the combination of theoretical understanding, algorithmic innovations, and experimental advancements leads to the realization of fault-tolerant quantum computation, revolutionizing many fields from science to industry. On this journey, the creativity and perseverance of individual researchers, demonstrating that large-scale theoretical discoveries can be made even with limited resources, will continue to inspire.

Open Science Articles (OSAs)

**Not:** The references provided here correspond to the first part of a prior study and are therefore not numbered or ordered based on their appearance in this paper. Additional citations introduced after the original submission are also omitted from the main reference list for consistency [322–467].

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